QUANTUM FIELD THEORIES ON ALGEBRAIC CURVES[1]

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ABSTRACT

In this talk the main features of the operator formalism for the $b-c$ systems on general algebraic curves developed in refs. [1]–[2] are reviewed. The first part of the talk is an introduction to the language of algebraic curves. Some explicit techniques for the construction of meromorphic tensors are explained. The second part is dedicated to the discussion of the $b-c$ systems. Some new results concerning the concrete representation of the basic operator algebra of the $b-c$ systems and the calculation of divisors on algebraic curves have also been included.

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1. INTRODUCTION

In this talk some aspects of quantum field theories on Riemann surfaces (RS) represented as algebraic curves are discussed. The latter are defined as $n-$sheeted branched covers of the complex sphere $\mathbf{CP}^1$. Any compact and orientable Riemann surface can be represented in this way [3]. Until now, most of the physical literature deals with the cases of hyperelliptic or Z_n symmetric curves (see e.g. [4]), while our curves are general. One advantage of working on algebraic curves is the possibility of constructing meromorphic tensors with poles at given points in an explicit way. For this reason and also in view of possible future applications in string theories and N=2 supersymmetric Yang–Mills field theories [5], the techniques for the derivation of meromorphic tensors on general algebraic curves are explained in Section 2 in great detail. In Section 3 we explain the operator formalism developed in refs. [1]–[2] for the $b-c$ systems [6]. The basis used here for expanding the $b-c$ fields on RS are similar to multipoint generalizations [7] of the Krichever–Novikov bases [8]. On the other hand, our formalism has some analogies with that of ref. [9]. It is understood that the parametrization of the moduli space of RS is given by the branch points of the curve. This choice entails some limitations, for instance in dealing with fermions [2] or if the entire moduli space of the RS of a given genus g should be considered. Nevertheless, such limitations are compensated by several advantages. First of all, the explicitness with which analytic field theories can be treated. For instance, the meromorphic tensors involved in the correlation functions of the $b-c$ fields can simply be written as rational functions of the coordinates describing the algebraic curve. Moreover, it is possible to treat field theories on $n-$sheeted branched covers of $\mathbf{CP}^1$ as multivalued field theories on the complex sphere. In our operator formalism, for example, all the correlation functions of the $b-c$ systems are computed exploiting simple normal ordering rules between operators defined on $\mathbf{CP}^1$. A concrete representation of the Fock space is provided in terms of semiinfinite forms at the end of Section 3. Another important point is that the multivalued field theories corresponding to the $b-c$ systems defined on algebraic curves can be reformulated as nonstandard conformal field theories on the complex sphere. For some simple curves, these conformal field theories have been explicitly constructed together with their primary fields, see refs. [1]–[2] and references therein. It has been shown that they obey a complicated nonabelian statistics. Furter possible developments of our results will be briefly discussed in the Conclusions.

[2] An exception is provided by the hyperelliptic curves.

2. THE LANGUAGE OF ALGEBRAIC CURVES

In order to fix the notations, let $\mathbf{C}$ be the complex plane and $\mathbf{CP}^1$ the complex projective line which coincides with the sphere or, equivalently, with the compactified complex plane $\mathbf{C}\cup\{\infty\}$. In this talk, we will consider RS as $n-$sheeted branched covers of $\mathbf{CP}^1$ [3]. The latter are algebraic curves defined as the locus of points $(z,y)\in\mathbf{CP}^1\otimes\mathbf{CP}^1$ for which the following equation is satisfied:

$$F(z,y)=0 \tag{2.1}$$

Here $F(z,y)$ denotes the so-called Weierstrass polynomial (WP), which is of the form:

$$F(z,y)=P_n(z)y^n+P_{n-1}(z)y^{n-1}+\ldots+P_1(z)y+P_0(z)=0 \tag{2.2}$$

with $P_s(z)=\sum\limits_{m=0}^{n_s}\alpha_{s,m}z^m$ for $s=0,\ldots,n$ and $n_s\in\mathbf{N}$. By a well known theorem, any Riemann surface can be expressed as an algebraic curve of this kind. Thus, from now on, we will use the words RS, $n-$sheeted branched covers of $\mathbf{CP}^1$ and algebraic curves interchangeably, though this language is somewhat imprecise. The best known algebraic curves are the hyperrelliptic curves, whose WP is simply given by: $y^2=-P_0(z)$. Also the slightly more general Z_n symmetric curves will be often mentioned here. They are characterized by a WP of the kind $y^n=-P_0(z)$.

At this point, we solve eq. (2.1) with respect to y. The resulting function $y(z)$ is multivalued on $\mathbf{CP}^1$ and exchanges its branches at a set of N_{bp} branch points $a_i,\ldots,a_{N_{np}}\in\mathbf{C}$ that will be defined below. To simplify our analysis, we assume that $P_n(z)=1$ in eq. (2.2). There is no loss of generality in this assumption. As a matter of fact, if $P_n(z)\neq 1$, we perform in eq. (2.1) the change of variables:

$$\tilde{y}(z)=y(z)P_n(z) \tag{2.3}$$

This does not affect the monodromy properties of $y(z)$, so that both $\tilde{y}(z)$ and $y(z)$ are meromorphic functions on the same Riemann surface. However, it is easy to realize that $\tilde{y}$ satisfies the equation $\sum_{i=0}^{n}\tilde{P}_{n-i}(z)\tilde{y}^{n-i}=0$, where now $\tilde{P}_n(z)=1$ and $\tilde{P}_{n-i}(z)=P_{n-i}(z)P_n^{i-1}(z)$ for $i=1,\ldots,n$.

Further, we suppose that none of the branch points is located at in $z=\infty$. The presence of a branch point at infinity can be detected trying the ansatz

$$y(z)\underset{z\to\infty}{\sim}\gamma z^p+\text{lower order terms} \tag{2.4}$$

in eq. (2.1) for large values of z. Solving eq. (2.1) at the leading order, one determines the values of γ and p. A branch point at $z = \infty$ is indicated by noninteger values of p. If this is the case, it is always possible without any loss of generality to perform a birational transformation on the curve of the following kind. First of all, the branch point at infinity is moved to a finite region of the plane by exploiting an $SL(2, \mathbf{C})$ transformation in z. Of course, in doing this the condition $P_n(z')$ is spoiled in the new variable z', but can easily be restored with the aid of the transformations (2.3) in y.

We are now ready to define the branch points $a_1, \ldots, a_{N_{bp}}$. Supposing that the curve (2.1) is nondegenerate[3], they are the solutions of of the following system of equations:

$$F(z, y) = F_y(z, y) = 0 \tag{2.5}$$

where $F_z(z, y) = dF(z, y)/dz$. It is useful to eliminate from eq. (2.5) the variable y. As an upshot, one obtains a polynomial equation in z of the kind $r(z) = 0$. Apart from very special curves, in which $r(z)$ has multiple roots, its degree coincides with the number of branch points N_{bp}. Let us notice that it is possible to derive the resultant $r(z)$ of eqs. (2.5) explicitly using the dialitic method of Sylvester, ref. [10], Vol. II, pag. 79. To each branch point a_i one can associate an integer ν_i, called the ramification index and defined as the number of branches of $y(z)$ that are exchanged at that branch point. Clearly $\nu_i \leq n$. At a branch point of ramification index ν_i, the WP $F(z, y)$ vanishes together with its first $n - 1$ partial derivatives in y. The genus g of the Riemann surface (2.1), the ramification indices of the branch points and the number of sheets n composing the curve are related together by the Riemann–Hurwitz formula:

$$2g - 2 = -2n + \sum_{s=1}^{L} (\nu_s - 1) \tag{2.6}$$

The genus can be explicitly computed once the form of the Weierstrass polynomial is known exploiting the Baker's method, see ref. [11], Vol. I, pag. 404 and will not be reported here.

On a Riemann surface S represented as an $n-$sheeted cover of $\mathbf{CP}^1$ there is a "canonical" complex structure inherited from $\mathbf{CP}^1$. A possible atlas on S is the following. Let us put $R = \max|a_i|$ and $\rho = \min|a_i - a_j|$ for $i, j = 1, \ldots, N_{bp}$. Near a branch point a_i of ramification index ν_i, or more precisely in the open disk $|z - a_i| < \rho$, we choose the local coordinate $\xi^{\nu_i} = z - a_i$. For $|z| > R$, the local coordinate is $z' = 1/z$. Let us notice that

[3] See for instance ref. [2] for the definition of nondegeneracy.

on the algebraic curve the set $|z| > R$ corresponds to an union of n disjoint discs. On the remaining open sets that build the covering of S the local coordinate is z.

To conclude this Section, we discuss the meromorphic tensors and their divisors. In particular, we are interested in tensors of the kind $T^{(l)}(z)dz^\lambda$, with λ upper or lower indices depending on the sign of $\lambda = 0, \pm 1, \pm 2, \ldots$ The meromorphic functions correspond to the case $\lambda = 0$. The branch index l is to recall that a tensor T is in general multivalued on $\mathbf{CP}^1$ due to its dependence in $y(z)$. Let $[D]$ a finite collections of points $p_1, \ldots, p_r$ with multiplicities $l_1, \ldots, l_r$ (integers) on the RS. $[D]$ is a so-called divisor [12]. To any meromorphic tensor Tdz^λ with zeros z_r of order k_r and poles p_s of order l_s, one can associate a divisor $[T]$ of the kind:

$$[T] = \sum_r k_r z_r - \sum_s l_s p_s \tag{2.7}$$

The most general tensor on an algebraic curve can be written in the form:

$$T^{(l)}(z)dz^\lambda = Q(z, y^{(l)}(z)) \frac{dz^\lambda}{[F_y(z, y^{(l)}(z))]^\lambda} \tag{2.8}$$

where $Q(z, y)$ is a rational function of z and y. The reason for which the factor $[F_y(z, y^{(l)}(z))]^{-\lambda}$ has been singled out in (2.8) will be clear below. From eq. (2.8) it is evident that, in order to construct tensors on an algebraic curve with poles and zeros at given points, it is necessary to know at least the divisors of the basic building blocks dz, y and $F(z, y)$. This can be done quite explicitly for general WP (2.2) if $P_n(z) = 1$ and there are no branch points at infinity. We only need the additional assumption that the polynomials $P_1(z)$ and $P_0(z)$ appearing in the WP have no roots in common. In this way, eq. (2.1) is approximated for small values of y by the relation $y \sim -P_0(z)/P_1(z)$. Therefore, the zeros $q_1, \ldots, q_{n_0}$ of $y(z)$ occur for values of z corresponding to the roots of $P_0(z)$. To study the behavior of $y(z)$ at infinity we try the ansatz (2.4) in eq. (2.1). If we retain only the leading order terms of $y(z)$ and of the polynomials $P_s(z)$ appearing in the WP (2.2), then eq. (2.1) is approximated by:

$$\gamma^n z^{pn} + \alpha_{s,n_s} \ldots + \gamma^{n-s} z^{p(n-s)+n_s} + \ldots + \alpha_{0,n_0} z^{n_0} = 0 \tag{2.9}$$

Since $y(z)$ is not branched at infinity, there should be n different solutions for γ that satisfy (2.9). Clearly, this can be true only if the first and last monomials $\gamma^n z^{pn}$ and $\alpha_{0,n_0} z^{n_0}$ entering in eq. (2.9) are of the same order near $z = \infty$, i. e. $z^{pn} \sim z^{n_0}$. Moreover, all the

other monomials must not contain higher order powers in z. Thus, we obtain for p the following result:

$$p = \frac{n_0}{n} = 1, 2, \dots \tag{2.10}$$

so that n_0 is an integer multiple of n. In this way we have derived the divisor of y:

$$[y] = \sum_{r=1}^{n_0} q_r \ - \ \sum_{j=0}^{n-1} \frac{n_0}{n} \infty_j \tag{2.11}$$

where the symbols ∞_j denote the points on the curve corresponding to $z = \infty$. As we see from the above equation, the degree of the divisor $[y]$ is zero as expected for a meromorphic function. Analogously, it is possible to compute also the divisors of $F_y(z, y)$ and dz:

$$[F_y] = \sum_{r=1}^{n_{bp}} (\nu_r - 1) a_r \ - \ (n-1) \sum_{j=0}^{n-1} \frac{n_0}{n} \infty_j \tag{2.12}$$

$$[dz] = \sum_{r=1}^{n_{bp}} (\nu_r - 1) a_r \ - \ 2 \sum_{j=0}^{n-1} \infty_j \tag{2.13}$$

The details are explained in ref. [2]. Exploiting the above divisors, one is able to prove that, if $\lambda \geq 2$, the following tensor has only a single pole at the point $z = w$ on the sheet $l = l'$:

$$K_\lambda^{(ll')}(z, w) dz^\lambda = \frac{1}{z - w} \frac{F(w, y^l(z))}{y^l(z) - y^{l'}(w)} \frac{dz^\lambda}{[F_y(z, y^l(z))]^\lambda} \tag{2.14}$$

where the indices l nd l' label the branches in z and w respectively. The tensor $K_\lambda^{(ll')}(z, w) dz^\lambda$ will be hereafter called the Weierstrass kernel (WK). If $\lambda = 1$, it is easy to check that

$$\omega_{ww'}^{ll'l''}(z) dz = K_1^{(ll')}(z, w) dz - K_1^{(ll'')}(z, w') dz \tag{2.15}$$

is a differential of the third kind with two poles in $z = w$ and $z = w'$ on the sheets $l = l'$ and $l = l''$ respectively. Let us remember that the differentials of the first kind are the holomorphic one forms, the differentials of the second kind are meromorphic one forms with no residua and the differentials of the third kind are meromorphic one forms with only single poles. Finally, a nondegenerate metric on the curve is $g_{z\bar{z}} dz d\bar{z} = (1 + z\bar{z})^\beta \frac{dz d\bar{z}}{|F_y(z, y)|^2}$ $\beta = p(n-1) - 2$.

To fix the ideas, in the next Section we will consider only the class of curves in which the degree n_s of the polynomials $P_s(z)$ appearing in eq. (2.1) is $n_s = n - s$. Let us denote these algebraic curves with the symbol Σ_g. Their genus g is given by the formula:

$$g = \frac{(n-1)(n-2)}{2} \tag{2.16}$$

Moreover $N_{bp} = 2g + 2(n-1)$ denotes the total number of simple $\nu_i = 2$ branch points. Finally, the divisors of dz, F_y and y correspond to the subcase $n_0/n = 1$ of eqs. (2.11)–(2.12).

3. THE OPERATOR FORMALISM FOR THE $b-c$ SYSTEMS

On the curves Σ_g defined above let us consider the theory of the $b-c$ systems with spin λ. If $\xi, \bar{\xi} \in \Sigma_g$ is a set of complex coordinates on the Riemann surface, the variable $z \in \mathbf{CP}^1$ can be regarded as a mapping $z : \xi \to \mathbf{CP}^1$. Thus, putting $\bar{\partial} \equiv \partial/\partial\bar{z}$, $b \equiv b^{(l)}(z(\xi), \bar{z}(\bar{\xi}))dz^\lambda$ and $c \equiv c^{(l)}(z(\xi), \bar{z}(\bar{\xi}))dz^{1-\lambda}$, it is possible to write the action of the $b-c$ systems as follows:

$$S_{\rm bc} = \int_{\Sigma_g} d^2z(\xi)\left(b\bar{\partial}c + \bar{b}\partial\bar{c}\right) \tag{3.1}$$

¿From now on, we will suppose that $\lambda \geq 2$. The case $\lambda = 1$ is complicated due to the presence of extra zero modes[4] in the c fields and will not be discussed here. A detailed treatment of the operator formalism with $\lambda = 1$ can be found in the second of refs. [1].

To expand the classical $b-c$ fields, we use the generalized Laurent series (GLS) of ref. [2]. These consist in two different expansions for the fields b and c and necessarily contain modes which are multivalued on $\mathbf{CP}^1$:

$$f_{k,i}(z) = \frac{z^{-i-\lambda}y^{n-1-k}(z)dz^\lambda}{(F_y(z,y(z)))^\lambda} \tag{3.2}$$

$$\phi_{l_i}(w)dw^{1-\lambda} = \frac{w^{-i+\lambda-1}dw^{1-\lambda}}{(F_y(w,y(w)))^{1-\lambda}}\times$$
$$\left(y^l(w) + y^{l-1}(w)P_{n-1}(w) + y^{l-2}(w)P_{n-2}(w) + ... + P_{n-l}(w)\right) \tag{3.3}$$

In ref. [2], we have proved that any tensor (2.8) is a linear combination of the modes (3.2)-(3.3). Moreover, after exchanging λ with $1-\lambda$, the two bases (3.2) and (3.3) turn

[4] By zero modes we intend the global solutions of the classical equations of motion.

out to be equivalent, i.e. the modes of the first can be expressed in terms of the modes of the second. The GLS for the $b-c$ fields read:

$$b(z)dz^{\lambda}=\sum_{k=0}^{n-1}b_k(z)dz^{\lambda}\qquad\qquad c(z)dz^{1-\lambda}=\sum_{k=0}^{n-1}c_k(z)dz^{1-\lambda}\tag{3.4}$$

where

$$b_k(z)dz^{\lambda}=\sum_{i=-\infty}^{\infty}b_{k,i}f_{k,i}(z)dz^{\lambda}\tag{3.5}$$

$$c_k(z)dz^{1-\lambda}=\sum_{i=-\infty}^{\infty}c_{k,i}\phi_{k,i}(z)dz^{1-\lambda}\tag{3.6}$$

The fields $b_k(z)dz^{\lambda}$ and $c_k(z)dz^{1-\lambda}$ have a physical significance. After quantizing the theory, in fact, we will see that fields with different values of k do not interact. This splitting of the $b-c$ systems into n different "$k-$sectors" is already evident at the classical level. For instance, it is possible to find a number N_{b_k} of b zero modes $\Omega_{k,i}(z)dz^{\lambda}$ that are proportional to $f_k(z)dz^{\lambda}$. The zero modes are of the form

$$\Omega_{k,i}dz^{\lambda}=f_{k,i}(z)dz^{\lambda}\tag{3.7}$$

The range of i in the above equation and the value of N_{b_k} strongly depend on the WP. The complete list of cases has been worked out in ref. [2] and will not be reported here. For instance, if $n>4$ and $\lambda>1$ we have:

$$\begin{cases} k=0,\ldots,n-1 \\ \lambda(2-n)+n-1-k\leq i\leq-\lambda \\ N_{b_k}=\lambda(n-3)+k-n+2 \end{cases}\tag{3.8}$$

It is possible to verify that $\sum\limits_{k=0}^{n-1}N_{b_k}=(2\lambda-1)(g-1)$ as desired. An important motivation for choosing the GLS (3.4)-(3.6) is that the WK (2.14) has a very simple expansion in the modes (3.2) and (3.3):

$$K_{\lambda}(z,w)=\frac{1}{z-w}\sum_{k=0}^{n-1}\phi_{k,1-\lambda}(w)f_{k,\lambda}(z)\tag{3.9}$$

At this point we quantize the $b-c$ systems on the algebraic curve, treating the theory as a set of n noninteracting field theories on $\mathbf{CP}^1$. To this purpose, we treat the coefficients $b_{k,i}$

and $c_{k,i}$ appearing in the GLS as quantum operators, for which we postulate the following commutation relations (CR):

$$\{b_{k,j}, c_{k',j'}\} = \delta_{kk'}\delta_{j+j',0}. \tag{3.10}$$

The consistency of the above assumptions, suggested by the multivaluedness of the modes (3.2)-(3.3) entering in the GLS (3.4), will be proved "a posteriori" by showing that in this way the correct correlations functions of the $b-c$ systems are obtained. The operators carrying the index k for $k = 0, \ldots, n-1$, act on the vacua $|0>_k$, where $|0>_k$ is the standard $SL(2, \mathbf{C})$ invariant vacuum of the complex sphere. The "total vacuum" of the $b-c$ systems is

$$|0\rangle = \otimes_{k=0}^{n-1}|0\rangle_k \tag{3.11}$$

Generalizing the usual definitions of creations and destruction operators at genus zero, we demand that:

$$b^-_{k,i}|0\rangle_k \equiv b_{k,i}|0\rangle_k = 0 \qquad \begin{cases} k = 0, \ldots, n-1 \\ i \geq 1-\lambda \end{cases} \tag{3.12}$$

$$c^-_{k,i}|0\rangle_k \equiv c_{k,i}|0\rangle_k = 0 \qquad \begin{cases} k = 0, \ldots, n-1 \\ i \geq \lambda \end{cases} \tag{3.13}$$

Moreover. we introduce the "out" vacua ${}_k\langle 0|$ requiring that

$${}_k\langle 0|b^+_{k,i} \equiv {}_k\langle 0|b_{k,i} = 0 \qquad \begin{cases} k = 0, \ldots, n-1 \\ i \leq -\lambda - N_{b_k} \end{cases} \tag{3.14}$$

$${}_k\langle 0|c^+_{k,i} \equiv {}_k\langle 0|c_{k,i} = 0 \qquad \begin{cases} k = 0, \ldots, n-1 \\ i \leq \lambda - 1 \end{cases} \tag{3.15}$$

¿From the above equations, we see that some of the $b_{k,j}$'s correspond to zero modes and the remaining ones are organized in two sets of creation and annihilation operators. The same applies to the $c_{k,j}$ with the only difference that there are no zero modes for them. ¿From the above equations and (3.9) we deduce the following natural definition of the "normal ordering" between the fields:

$$b_k(z)c_k(w)dz^\lambda dw^{1-\lambda} =: b_k(z)c_k(w) : dz^\lambda dw^{1-\lambda} + K_\lambda(z,w)dz^\lambda dw^{1-\lambda} \tag{3.16}$$

The "time ordering" is implemented by the requirement that the fields $b(z)$ and $c(w)$ are radially ordered with respect to the variables z and w. Let us notice that, remarkably, there is no need of introducing more complicate time ordering which is sensitive to the branches of the $b-c$ fields. Finally, in order to take into account the zero modes, we impose the following conditions:

$$ {}_k\langle 0|0\rangle_k = 0 \qquad \text{if } N_{b_k} \neq 0; \qquad\qquad {}_k\langle 0|\prod_{i=1}^{N_{b_k}} b_{k,i}|0\rangle_k = 1. \tag{3.17} $$

At this point, we have all the ingredients to compute the correlation functions of the $b-c$ systems within our operatorial formalism. Here we report only the result for the propagator in the case of $n>4$, $\lambda>1$:

$$ \frac{\langle 0|b^{(l)}(z)c^{(l')}(w)\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle}{\langle 0|\prod_{I=1}^{N_b} b^{(l_I)}(z_I)|0\rangle} = $$

$$ \frac{\det\begin{vmatrix} \Omega^{(l)}_{1,1}(z) & \dots & \Omega^{(l)}_{n-1,N_{b_{n-1}}}(z) & K^{(ll')}_\lambda(z,w) \\ \Omega^{(l_1)}_{1,1}(z_1) & \dots & \Omega^{(l_1)}_{n-1,N_{b_{n-1}}}(z_1) & K^{(l_1l')}_\lambda(z_1,w) \\ \vdots & \ddots & \vdots & \vdots \\ \Omega^{(l_{N_b})}_{1,1}(z_{N_b}) & \dots & \Omega^{(l_{N_b})}_{n-1,N_{b_{n-1}}}(z_{N_b}) & K^{(l_{N_b}l')}_\lambda(z_{N_b},w) \end{vmatrix}}{\det|\Omega_I(z_J)|} \tag{3.18} $$

In the above equation, N_b denotes the total number of b zero modes. Let us notice that the right hand side of eq. (3.18) is a ratio of correlators containing multivalued fields on thecomplex plane, whereas the left hand side represents the propagator of the $b-c$ systems on the algebraic curve. The detailed computations of the general $n-$ point functions of the $b-c$ systems can be found in refs. [1]–[2].

To conclude this Section, some remarks should be made about the structure of the linear space on which the $b-c$ systems are realized as quantum field theories. A concrete realization of the representation space can be given in terms of ref. semiinfinite forms via the identifications

$$ b_{k,j} \hookrightarrow \beta_{k,j}\wedge ... \qquad\qquad c_{k,j} \hookrightarrow \frac{\partial}{\partial\beta_{k,-j}} \tag{3.19} $$

The "in" vacuum state is filled with the excitations β_{k,p_k} with $p_k \geq 1-\lambda$. For

$$ |x\rangle = \beta_{k,j_1}\wedge\beta_{k,j_2}\wedge ... \tag{3.20} $$

and

$$|y\rangle = \beta_{k,j'_1} \wedge \beta_{k,j'_2} \wedge ... \tag{3.21}$$

one introduces the bilinear form

$$\langle y|x\rangle \equiv ... \wedge \beta_{k,-j'_2+H} \wedge \beta_{k,-j'_1+H} \wedge \beta_{k,j_1} \wedge \beta_{k,j_2} \wedge ... \tag{3.22}$$

where

$$H = 1 - 2\lambda - N_{b_k} \tag{3.23}$$

The above bilinear form is by definition equal zero unless all the excitations are present at the RHS of (3.22). If not zero, it can take only the values ± 1 (some sign convention has to be made). The vacuum state and the bilinear forms introduced above satisfy the requirements (3.17). In the linear space of states

$$b_{k_1,s_1} ... c_{l_1,t_1} ... |0\rangle \tag{3.24}$$

it is natural to introduce another bilinear form

$$(y|x) \equiv ... \wedge \beta_{k,-j'_2+H} \wedge \beta_{k,-j'_1+H} \wedge \beta_{k,j_1} \wedge \beta_{k,j_2} \wedge ... \tag{3.25}$$

by demanding that (3.25) is zero unless all the excitations except from those corresponding to zero modes are present on the RHS. This time $(0|0) = 1$. We do not obtain a Hilbert space of states as $(\cdot|\cdot)$ is not positive definite. This should not be a surprise as for instance the $b-c$ systems for $\lambda = 2$ are the Faddeev-Popov ghosts for reparameterization invariance in string theory. Let us mention also that the consistency of the above representation requires the following reality conditions for the elementary excitations:

$$b^+_{k,j} = b_{k,-j+H}, \qquad c^+_{k,j} = c_{k,-j+H} \tag{3.26}$$

It is important that the set of $b_{k,j}$ corresponding to zero modes of the theory remains invariant under the conjugation introduced in (3.26).

4. CONCLUDING REMARKS

As we have seen, any meromorphic tensor on general algebraic curves can be expanded as linear combinations of the multivalued modes (3.2)–(3.3). The operator formalism, instead, has been tested until now only in the case of the $b-c$ systems. However, one should try to generalize it also to other field theories. For instance, the $\beta-\gamma$ systems seems to be treatable as well. Also the theory of massless scalar systems is a good candidate for applying our methods, but the fact that the correlation functions are no longer meromorphic complicates the study of this case. Finally, we hope that our formalism could be useful in discussing the minimal models on algebraic curves [13].

References

[1] F. Ferrari, J. Sobczyk and W. Urbanik, *Jour. Math. Phys.* **36** (1995), 3216; F. Ferrari and J. Sobczyk, *Int. Jour. Mod. Phys.* **A11** (1996), 2213.

[2] F. Ferrari and J. Sobczyk, *Journal Geom. Phys.* **19** (1996), 287.

[3] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York 1978.

[4] D. Lebedev and A. Morozov, *Nucl. Phys.* **B302** (1986), 163; E. Gava, R. Iengo and G. Sotkov, *Phys. Lett.* **207**B (1988), 283; A. Yu. Morozov and A. Perelomov, *Phys. Lett.* **197**B (1987), 115; M. A. Bershadsky and A. O. Radul, *Int. Jour. Mod. Phys.* **A2** (1987), 165.

[5] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994), 19.

[6] D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986), 93.

[7] R. Dick, *Fortschr. Physik* **40** (1992), 519; M. Schlichenmaier, *Lett. Math. Phys.* **19** (1990), 151.

[8] I. M. Krichever and S. P. Novikov, *Funk. Anal. Pril.* **21** No.2 (1987), 46; **21** No.4 (1988), 47; J. Geom. Phys. **5** (1988) 631.

[9] L. Bonora, A. Lugo, M. Matone and J. Russo, *Comm. Math. Phys.* **123** (1989), 329.

[10] F. Enriques and O. Chisini, *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, Zanichelli, Bologna (in italian).

[11] A. R. Forsyth, *Theory of Functions of a Complex Variable*, Vols I and II, Dover Publications, Inc., New York, 1965.

[12] H. Farkas and I. Kra, *Riemann Surfaces*, Springer Verlag, 1980

[13] C. Crnkovic, G.M. Sotkov and M. Stanishkov, *Phys. Lett.* **220**B (1989), 397; S. A. Apikyan and C. J. Efthimiou, *Minimal Models of CFT on Z_N Surfaces*, hep-th 9610051.